

# Competitive Equilibrium in Two Sided Matching Markets with General Utility Functions

Saeed Alaei<sup>\*</sup>; Kamal Jain<sup>†</sup>; Azarakhsh Malekian<sup>‡</sup>

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## Abstract

In this paper, we study the class of competitive equilibria in two sided matching markets with general (non-quasilinear) utility functions. Mechanism design in general non-quasilinear setting is one of the biggest challenges in mechanism design. General non-quasilinear utilities can for example model smooth budget constraints as a special case. Due to the difficulty of dealing with arbitrary non-quasilinear utilities, a large fraction of the existing work have considered the simpler case of quasilinear utilities with hard budget constraints and they all rely on some form of ascending auction. For general non-quasilinear utilities, we show that such ascending auctions may not even converge in finite time. As such, almost all of the existing work on general non-quasilinear utility function ([4, 6, 17]) have resorted to non-constructive proofs based on fixed point theorems or discretization. In this paper, we give the first direct characterization of competitive equilibria in such markets. Our approach is constructive and solely based on induction. Our characterization reveals striking similarities between the payments at the lowest competitive equilibrium for general utilities and VCG payments for quasilinear utilities. We also show that the mechanism that outputs the lowest competitive equilibrium is group strategyproof. We also present a class of price discriminating truthful mechanisms for selling heterogeneous goods to unit-demand buyers with general utility functions and from that we derive a natural welfare maximizing mechanism for ad-auctions that combines pay per click and pay per impression advertisers with general utility functions. Our mechanism is group strategyproof even if the search engine and advertisers have different estimates of clickthrough rates. This also answers an open question raised by [1].

## 1 Introduction

In this paper, we study the class of competitive equilibria in two sided matching markets with general utility functions. In these markets, agents form a one-to-one matching and monetary transfers are made between matched agents. Utility of each agent is a function of whom she is matched to and the amount of the monetary transfer to/from her partner. For the most of this paper, we work with simpler markets consisting of a set of buyers and a set of heterogeneous good. In section 5, we show that the more general model can be reduced to this simpler buyer/good model. We assume that the utility of each buyer depends on the choice of good she receives and

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<sup>\*</sup>Dept. of Computer Science, University of Maryland, College Park, MD 20742. saeed@cs.umd.edu. Part of this work was done when the author was visiting Microsoft Research, Redmond. This work was partially supported by the NSF grant CCF-0728839

<sup>†</sup>Microsoft Research, Redmond, WA 98052. kamalj@microsoft.com.

<sup>‡</sup>Dept. of Computer Science, Northwestern University, Evanston, IL 60208. a-malekian@northwestern.edu.

the price she pays but it is not necessarily a quasi-linear function of the payment. Non-quasilinear utilities can be used for example to model smooth budget constraints.

A competitive equilibrium, in these markets, is essentially an assignment of prices to goods together with a feasible allocation of goods to buyers such that every buyer receives her most preferred good at the announced prices and every unallocated good has a price of 0. This is also referred to as an envy-free equilibrium for the buyer/good model. In the case of unit-demand buyers, each buyer would be allocated at most a single good. With quasi-linear utilities, buyer  $i$ 's utility for good  $j$  as a function of payment can be written as  $u_i^j(x) = v_i^j - x$  where  $v_i^j$  is the valuation of buyer  $i$  for good  $j$  and  $x$  is the payment. In this case, social welfare is well-defined and VCG is applicable. The efficient allocation can be computed using a maximum weight matching on the bipartite graph consisting of buyers/goods with the edge between buyer  $i$  and good  $j$  having a weight of  $v_i^j$ . The VCG payoffs/payments would then correspond to a minimum weighted cover on this graph [15]. For general utilities, the functions  $u_i^j(x)$  could be any continuous decreasing function of  $x$ . In this case, social welfare is not well defined and VCG is not applicable.

As a motivating example of a unit-demand market with non-quasi-linear utilities, consider a housing market in which each seller owns a house and each buyer wants to buy a house. Typically, a buyer will have a smooth budget constraint. For example, they may need to get a loan/mortgage to pay for the house and so the actual cost will include interests, fees, etc. in addition to the actual payment made to the seller. This cost may depend on the choice of the house as well (e.g., the interest rate may depend on the condition of the house). With non-quasilinear utilities, this can be modeled as  $u_i^j(x) = v_i^j - c_i^j(x)$  in which  $c_i^j(x)$  is the cost as a function of the price of good  $j$ .

## 2 Related Work

In the abstract mathematical form, the problem we are looking at is a one-to-one matching with monetary transfers and general utilities as described by Demange and Gale[4]. In this model, the set of competitive equilibria corresponds exactly to the outcomes that are in the core. Demange and Gale also proved the lattice structure on the set of competitive equilibria although the lattice structure was already discovered by Shapley and Shubik [19] for the case of quasilinear utilities. Demange, Gale and Sotomayor [5] proposed an ascending auction for the quasilinear setting to compute a competitive equilibrium. The existence of competitive equilibria for general utilities was proved by Quinzii [17]. Quinzii showed that the game defined by this model is a “*Balanced Game*” and for general  $n$ -person balanced games it was already shown by Scarf [18] that the core is non-empty. Using a different method, Gale [6] showed that for a more general class of preferences (i.e., preferences are not even required to be monotone in payment, yet they should still satisfy some other milder conditions) a competitive equilibrium always exists. Gale’s proof is based on a generalization of the KKM lemma [12] which is the continuous variant of the Sperner’s lemma. Both of these proofs only show the existence of an equilibrium and are non-constructive. As such, they don’t help much in understanding the properties of the equilibria. <sup>1</sup>

Using a completely different approach, Kelso and Crawford [11] studied the equilibria of the more general case of many-to-one matching with monetary transfers using discretization, i.e. the prices are chosen from a discrete set rather than from a continuum). Their approach can be considered an extension of the deferred acceptance algorithm of Gale and Shapley for college admission and stable

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<sup>1</sup>Scarf’s proof actually provides an algorithm based on the pivoting algorithm of Lemke and Howson [14]. When combined with Quinzii’s construction, that would lead to a construction that requires  $2^{O(n!)}$  operations which runs on a matrix with  $O(n!)$  columns. Nevertheless, the resulting algorithm is more of an exhaustive search algorithm and does not provide any insight into the equilibrium structure.

marriage[7]. Kelso and Crawford state their problem in the context of matching workers to firms. They introduce the notion of “*Gross Substitutes*”(GS) and show that if firms’ preferences satisfy GS then the core is non-empty. Later on, Hatfield and Milgrom[10] presented a unified framework of many-to-one matching with contracts which subsumes the Kelso-Crawford model. Their approach is also based on discretization. They replace the finite set of discrete prices with a finite set of contracts where a contract could include any general term which may include a monetary transfer amount as well. They describe their model in the context of hospitals and doctors and show that if hospitals preferences over the set of possible contracts satisfy GS and also if doctors have strict preferences over the set of contracts then the core is non-empty. They show that the set of core outcomes form a lattice and that the infimum of the lattice correspond to the doctor optimal outcome while the supremum of the lattice correspond to the optimal outcome for the hospitals. They provide an iterative procedure for finding the core outcomes based on the discrete version of Tarski’s fixed point theorem. They also characterize another condition which they call the “*Law of Aggregate Demand*” under which the doctor optimal outcome is also group strategyproof for the doctors. Recently, Hatfield and Kominers [9] generalized this to many-to-many matchings with contracts.

Leonard [15] first showed that in one-to-one markets with quasilinear utilities, prices at the lowest competitive equilibrium equal VCG payments. Gul and Stacchetti[8] studied many-to-one matchings in the context of allocation of indivisible goods to consumers with quasi-linear utilities. Their model differs from the model of Kelso and Crawford in that they do not require discrete prices but instead require the utilities to be quasi-linear in money. They show the existence of competitive equilibria given that consumers’ preferences satisfy GS. They also show that not only is GS sufficient but it is also necessary. Similarly, Bikhchandani and Mamer [16] showed the existence of competitive equilibria for the same model but without indivisibility using a different approach. Their proofs crucially needs the quasilinearity of utilities and their approach cannot be extended to general utilities.

Ausubel and Milgrom [3] also studied the many-to-one matching in the the context of allocation of indivisible goods to consumers with quasi-linear utilities. They propose an ascending package auction to compute the outcome. They assume that all the goods are initially owned by one seller and as such the set of competitive equilibria is only a strict subset of the core. They consider the core outcomes and not just the competitive equilibria. They present an ascending package auction that always results in a core outcome even in the absence of GS preferences. They show that if consumers’ preferences satisfy GS then the outcome of their auction coincides with the VCG outcome. More specifically, they show that their auction precisely computes the VCG outcome whenever the VCG outcome is in the core. They also show that GS is the the necessary and sufficient condition for the VCG outcome to be in the core. Their auction, however, requires payments to be chosen from a finite discrete set. Their setting can be modeled as a special case of the Milgrom and Hatfield matching with contracts [10]. Their proofs also crucially depend on quasilinearity of utilities.

There are also related work that consider one-to-one matching markets with quasilinear utilities and hard budget constraint. Aggarwal et al.[1], consider this problem in the context of Ad-Auctions with advertisers having slot specific hard budget constraints. They prove the existence of a budget-feasible competitive equilibrium and present a truthful auction mechanism based on that. They present an extension of the Hungarian method[13] for computing the equilibrium and their proofs are based on this construction. Ashlagi et al. [2] also consider a similar problem but they assume a single hard budget constraint and a single value per click for each advertiser and require separable click through rates. In subsection 4.2, we discuss the major difficulty of dealing with soft budget constraints as opposed to hard budget constraints.

### 3 Our Contribution

In this paper we study the class of competitive equilibria in unit demand markets with general utility function in continuous setting. We must emphasize that all of the earlier works except for [4, 17, 6] either crucially require quasilinear utilities or work in a discrete setting. Our main contributions are the following:

- In Theorem 2, we present a construction using an inductive characterization of prices/payoffs at the competitive equilibria that reveals interesting similarities between VCG payments for quasilinear utilities and the prices at the lowest competitive equilibrium for general utilities. In Theorem 3 we give a simple proof for group strategyproofness based on a critical property of the lowset/highest competitive equilibria. All of the earlier works only proved the existence of a competitive equilibrium using either fixed point theorems or discretization without providing an exact characterization of the equilibria. We present a simple characterization that has a natural interpretation. Our characterization provides a deeper insight into the structure of the equilibria.
- In section 6, we suggest a mechanism for ad-auctions that can naturally combine both pay per click and pay per impression advertisers in a general setting in which advertisers could submit a separate utility function for each slot as a price of that slot. These utility functions could be any arbitrary function<sup>2</sup> of the price of that slot. Furthermore, our mechanism is group strategyproof even if search engine and advertisers have different estimates of clickthrough rates. This also answers an open question raised by [1]. Furthermore, our mechanism is welfare maximizing in the sense that it maximizes the combined welfare of the search engine and any group of advertisers who have quasilinear utilities and agree with the search engine on clickthrough rates and assuming that the search engine has the correct clickthrough rates. In particular, if all advertisers have quasilinear utilities and agree with the search engine on clickthrough rates, then the outcome of our mechanism coincides with the VCG outcome.

### 4 Model and Main Results

In this section, we consider competitive equilibria in two sided markets with goods on one side and buyers on the other side. Later in section 5, we consider the more general model with agents on both sides and show that it can be reduced to the simpler buyer/good model. In subsection 4.1, we formally define the problem and our notation. In subsection 4.2, we explain the main challenges of dealing with non-quasilinear utilities and explain why it is much harder to prove these results for non-quasilinear utilities compared to their quasilinear counterparts. In subsection 4.3, we present our main general theorems.

#### 4.1 Model

In this subsection, we formally define the problem and our notation.

We denote by  $M = (I, J, \{u_i^j\})$ , a market  $M$  with the set of unit demand buyers  $I$  and the set of goods  $J$  such that the utility of buyer  $i$  for receiving good  $j$  at price  $x$  is given by the monotonically decreasing function  $u_i^j(x)$  which is privately known by buyer  $i$ . We assume that for a large enough

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<sup>2</sup>It has to be continuous and decreasing in the price of the slot and should become non-positive for a high enough price.

$x$ ,  $u_i^j(x)$  becomes zero or negative<sup>3</sup>. We will use  $p_i^j(\cdot)$  to denote the inverse of  $u_i^j(\cdot)$ . Next, we formally define a *Competitive Equilibrium*.

**Definition 1** (Competitive Equilibrium). *Given a market  $M = (I, J, \{u_i^j\})$ , a “Competitive Equilibrium” of  $M$  is an assignment of prices to goods together with a feasible matching of goods to buyers such that each buyer receives her most preferred good at the assigned prices and every unmatched good has a price of 0. Formally, we say that  $W$  is a competitive equilibrium of  $M$  with price vector  $p(W)$  and payoff vector  $u(W)$  if and only if there exists a “Supporting Matching”  $\mu$  such that the following conditions hold. We use  $\mu(i)$  to denote the good that is matched to buyer  $i$ . Also, let  $\mathbf{p} = p(W)$  and  $\mathbf{u} = u(W)$ :*

$$\forall i \in I, \forall j \in J : \quad \begin{cases} \mathbf{u}_i = u_i^j(\mathbf{p}^j) & j = \mu(i) \\ \mathbf{u}_i \geq u_i^j(\mathbf{p}^j) & j \neq \mu(i) \end{cases} \quad (4.1)$$

$$\forall i \in I : \quad \mu(i) = \emptyset \Rightarrow \mathbf{u}_i = 0 \quad (4.2)$$

$$\forall j \in J : \quad \mu^{-1}(j) = \emptyset \Rightarrow \mathbf{p}^j = 0 \quad (4.3)$$

$$\forall i \in I, \forall j \in J : \quad \mathbf{u}_i \geq 0, \mathbf{p}^j \geq 0 \quad (4.4)$$

We denote an unmatched buyer or good by  $\mu(i) = \emptyset$  or  $\mu^{-1}(j) = \emptyset$ . We use  $\mu(W)$  to denote a supporting matching for  $W$ . Note that there could be more than one supporting matching for a given  $W$  so we assume  $\mu(W)$  may return any one of them. We will denote the set of all competitive equilibria for a market  $M$  by  $\mathcal{W}(M)$ .

## 4.2 The Main Challenges of Non-Quasilinear Utilities

In this subsection, we explain the main challenges of dealing with non-quasilinear utilities. First, it is helpful to explain the connection between VCG and competitive equilibria in unit-demand markets with quasilinear utilities.

VCG is based on maximizing the social welfare which is defined as the sum of the utilities. Taking the sum of quasilinear utility functions makes sense because they are measured in the same units and the payment terms cancel out. However, with general utilities, social welfare is not well-defined since different agents’ utilities are not measured in the same units and are therefore non-transferrable (i.e. transferring \$1 from one agent to another does not transfer the same amount of utility). In our problem, with quasilinear utilities, the utility functions would be of the form  $u_i^j(x) = v_i^j - x$  where  $v_i^j$  is the value of the agent  $i$  for good  $j$ . We could then construct a complete bipartite graph with agents and goods in which each edge  $(i, j)$  has a weight of  $v_i^j$ . A social welfare maximizing mechanism like VCG would pick a maximum weight matching in this graph which is also captured by the following LP:

$$\begin{array}{ll} \textbf{Primal:} & \max \cdot \sum_{i \in I} \sum_{j \in J} v_i^j \mathbf{x}_i^j \\ & \forall i \in I : \quad \sum_{j \in J} \mathbf{x}_i^j \leq 1 \\ & \forall j \in J : \quad \sum_{i \in I} \mathbf{x}_i^j \leq 1 \\ & \quad \quad \mathbf{x}_i^j \geq 0 \\ \textbf{Dual:} & \min \cdot \sum_{i \in I} \mathbf{u}_i + \sum_{j \in J} \mathbf{p}^j \\ & \forall i \in I, \forall j \in J : \quad \mathbf{u}_i + \mathbf{p}^j \geq v_i^j \\ & \quad \quad \mathbf{u}_i \geq 0 \\ & \quad \quad \mathbf{p}^j \geq 0 \end{array} \quad (4.5)$$

<sup>3</sup>This is to ensure that  $u_i^j(\cdot)$  is invertible. We also require the domain and the range of  $u_i^j(\cdot)$  to cover the whole  $\mathbb{R}$ . Since at an equilibrium both  $x$  and  $u_i^j(x)$  are positive, we can easily extend the domain of any  $u_i^j(\cdot)$  to the whole  $\mathbb{R}$  to meet this requirement.

Notice that there is a one-to-one correspondence between solutions of the dual program and the competitive equilibria (observe that by complementary slackness, if  $\mathbf{x}_i^j > 0$  then  $\mathbf{u}_i = v_i^j - \mathbf{p}^j$ ). It is not hard to show that the prices at the lowest competitive equilibrium (the one that has the lowest prices) correspond to the VCG payments. Furthermore, any competitive equilibrium of the market leads to a social welfare maximizing allocation (this follows from strong duality). To compute a maximum weight matching in this graph we can use the *Hungarian Method* [13]. Interestingly, the Hungarian method is equivalent to the following ascending price auction proposed by Demange, Gale and Sotomayor[5]:

**Definition 2** (Ascending Price Auction). *Set all the prices equal to 0. Find a minimally over demanded subset of goods, i.e. a subset  $T$  of goods such that there is a subset  $S$  of the buyers who strictly prefer the goods in  $T$  at the current prices and  $|S| > |T|$ . Increase the prices of goods in  $T$  at the same rate until one of the buyers in  $S$  becomes indifferent between a good outside of  $T$  and her preferred good in  $T$ . At that point, recompute the minimally over-demanded subset and repeat this process until there is no over demanded subset of goods.*

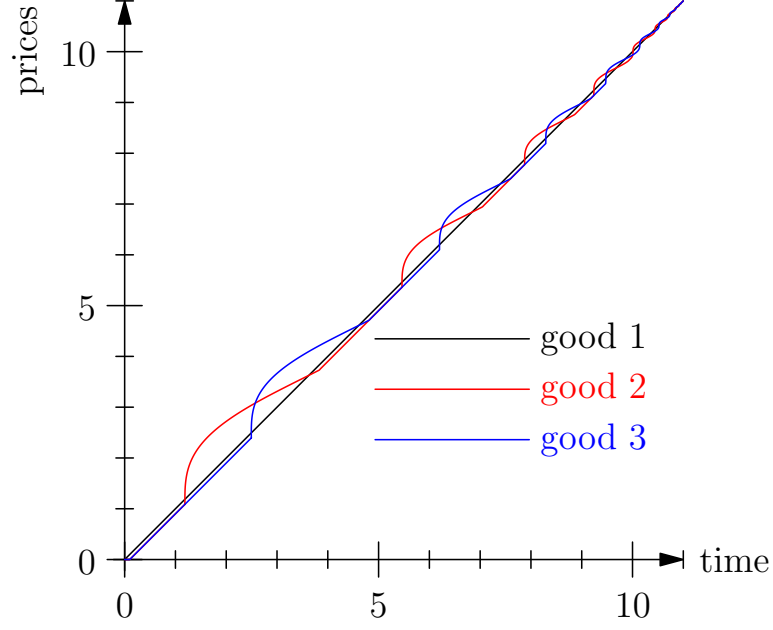
In fact, all of the existing methods for computing the lowest competitive equilibrium, that we are aware of, are based on running an ascending auction of the above form or a similar ascending auction. Furthermore, in all of the related work that are based on such ascending auctions, the proofs are heavily tied with the way the ascending auction proceeds and the fact that it stops in finite time. Essentially, all of these auctions work as follows: They advance the prices at some rate to the next point at which there is a change in demand structure. Then, they recompute the rates and repeat. For quasilinear utility functions, the ascending auction stop after  $O(|I||J|)$  iterations (Each time the combinatorial structure of the demand changes we start a new iteration).

Unfortunately, these ascending auctions may not even terminate in finite time if utilities are not quasilinear. The problem occurs when we try to raise the prices of goods in set  $T$ . With quasilinear utilities, when we raise the prices of all the goods in  $T$  at the same rate, for buyers in  $S$ , the relative preferences over the goods in  $T$  does not change. However, that is not true for general utility functions. For general utilities, we may need to raise the prices of goods in  $T$  at different and possibly non-constant rates and even then the preferences of buyers in  $S$  over goods in  $T$  may change an unbounded number of times. We demonstrate this in the following example:

**Example 1.** *Suppose there are 3 goods and 4 buyers with utility functions as given in the following table in which  $V \geq 2$  is some constant and  $x$  is the price of the corresponding good:*

	good 1	good 2	good 3
buyer 1	$V + 1 - x$	$V + 1 - x$	$V + 1 - x$
buyer 2	0	$V + 1 - x$	0
buyer 3	0	0	$V + 1 - x$
buyer 4	$V - x$	$V - c(x)$	$V - c'(x)$

All buyers have quasilinear utilities except buyer 4 for whom  $c(x) = x + \frac{V-x}{V} \sin(V \log(V - x))$  and  $c'(x) = x + \frac{V-x}{V} \cos(V \log(V - x))$ . Notice that both  $c(\cdot)$  and  $c'(\cdot)$  are strictly increasing in  $x$  if  $V \geq 2$ , so all utility functions are strictly decreasing in prices. Figure 1 shows the prices of goods during the ascending auction. We should emphasize that in this particular example, the ascending path of prices is unique. The ascending auction can only increase the prices of goods that are over demanded, i.e., demanded by at least two buyers. Furthermore, it can only raise the price of a good to the point where the demand of that good is about to drop to 1. Therefore, for every good with a positive price during the auction there should be at least a demand of 2. Observe that the demand set of buyer 1 and 4 changes an infinite number of times during the ascending auction. Specifically, the demand set of both buyer 1 and 4 include good 1 at all times. However, the demand set of buyer 1 includes good 2 and/or good 3 only at the times in which the price curves of those goods overlap with the



The blue and red curves have been slightly shifted down to make the black curve visible.

Figure 1: prices of goods in the ascending auction of Ex. 1 assuming that  $V = 11$  and that the price of good 1 is increased at the rate of 1 .

price curve of good 1. Similarly, the demand set of buyer 4 includes good 2 and/or good 3 only at the times in which the price curves of those goods do not overlap with the price curve of good 1. Observe that the demand structure changes an infinite number of times as the price of the goods approach  $V$ . So an ascending auction does not stop in finite time.

The previous example, although contrived, illustrates what could go wrong with ascending auctions and constructive proofs that are based on them. In general, ascending auctions are very sensitive to the structure of utility functions. Later, in Theorem 2, we present a direct way of computing the lowest competitive equilibrium without running an ascending auction.

**Hard budgets vs. Smooth budgets:** Notice that with hard budget constraints, utility functions are quasilinear except at the point where buyers hit their budget limits. Therefore, the issue that was outlined in Ex. 1 does not arise with quasilinear utilities and hard budget constraints. In fact, ascending auctions with hard budget constraints converge almost as fast as ascending auction with quasilinear utilities because each buyer may hit her budget limits at most  $|J|$  times (once per each good) and beyond that they never demand that good again. This is what makes general non-quasilinear utilities much harder to work with compared to quasilinear utilities with hard budgets. It is worth mentioning that the related work of [1] and [2] are based on such ascending auctions.

### 4.3 Main Results

In this subsection, we state our main theorems that capture the important properties of competitive equilibria. Our main contributions in this section are Theorem 2 which characterizes the prices/payoffs at the highest/lowest competitive equilibria and Theorem 3 that establishes the group strategyproofness of the mechanism that selects the lowest competitive equilibrium. Our charac-

terization reveals a deep connection between the way VCG computes its payments and the way prices can be computed at the lowest competitive equilibria. We start by showing that the set of competitive equilibria form a lattice.

**Theorem 1** (Equilibrium Lattice). *For a given market  $M = (I, J, \{u_i^j\})$ , with the set of competitive equilibria  $\mathbb{W}(M)$ , we define a partial ordering as follows. For any two competitive equilibria  $W, W' \in \mathbb{W}(M)$ , we say  $W \leq W'$  iff  $p(W) \leq p(W')$  (or equivalently  $u(W) \geq u(W')$ )<sup>4</sup>. The partially ordered set  $(\mathbb{W}(M), \leq)$  is a complete lattice. The inf and sup operators on the lattice are defined as follows. Let  $W_{\inf} = \inf(W, W')$  and  $W_{\sup} = \sup(W, W')$ . Both  $W_{\inf}$  and  $W_{\sup}$  are valid competitive equilibria (we provide a supporting matching for each one):*

$$W_{\inf} : \begin{cases} p^j(W_{\inf}) = \min(p^j(W), p^j(W')) \\ u_i(W_{\inf}) = \max(u_i(W), u_i(W')) \\ \mu_{\inf}(i) = \begin{cases} \mu(i) & \mathbf{u}_i \geq \mathbf{u}'_i \\ \mu'(i) & \mathbf{u}_i < \mathbf{u}'_i \end{cases} \end{cases} \quad W_{\sup} : \begin{cases} p^j(W_{\sup}) = \max(p^j(W), p^j(W')) \\ u_i(W_{\sup}) = \min(u_i(W), u_i(W')) \\ \mu_{\sup}(i) = \begin{cases} \mu(i) & \mathbf{u}_i < \mathbf{u}'_i \\ \mu'(i) & \mathbf{u}_i \geq \mathbf{u}'_i \end{cases} \end{cases} \quad (4.6)$$

*In particular, the lattice has a unique minimum which we refer to as the lowest competitive equilibrium (i.e. has the lowest prices and the highest payoffs) and a unique maximum which we refer to as the highest competitive equilibrium (i.e. has the highest prices and the lowest payoffs)*<sup>5</sup>.

Throughout the rest of the paper, we use the lattice structure of the set of competitive equilibria without making explicit references to Theorem 1.

Before we present our theorems, we define the following notation. Note that we can fully specify a competitive equilibrium  $W$  by just specifying either  $u(W)$  or  $p(W)$ . Given either the price vector or the payoff vector, we can compute the other one by taking the induced prices/induced payoffs as defined next.

**Definition 3** (Induced Payoffs  $u(\mathbf{p})$ , Induced Prices  $p(\mathbf{u})$ ). *We use  $u(\mathbf{p})$  to denote the “Induced Payoffs” of buyers from price vector  $\mathbf{p}$  which is the best payoff that each buyer can possibly get given the prices  $\mathbf{p}$ . Similarly, we use  $p(\mathbf{u})$  to denote the Induced Prices of goods from the payoff vector  $\mathbf{u}$ . The formal definition is as follows (remember that  $p_i^j(\cdot)$  is the inverse of  $u_i^j(\cdot)$ ):*

$$u_i(\mathbf{p}) = \max(\{u_i^j(\mathbf{p}^j) | j \in J\} \cup \{0\}) \quad (4.7)$$

$$p^j(\mathbf{u}) = \max(\{p_i^j(\mathbf{u}_i) | i \in I\} \cup \{0\}) \quad (4.8)$$

It is easy to see that if  $W$  is a competitive equilibrium then  $u(W) = u(p(W))$  and  $p(W) = p(u(W))$ <sup>6</sup>. Throughout this paper, we use bold letters  $\mathbf{p}$  and  $\mathbf{u}$  to denote variables representing price/payoff vectors and non-bold letters  $p$  and  $u$  to denote functions returning price/payoff vectors. The next theorem states the main result of this paper. In what follows,  $u_i(\mathbf{p})$  and  $p^j(\mathbf{u})$  denote the induced payoff and induced price as defined in (4.7) and (4.8) respectively.

**Theorem 2** (Inductive Equilibrium). *Given a market  $M = (I, J, \{u_i^j\})$ , a competitive equilibrium always exists. Furthermore, the lowest and the highest competitive equilibria can be computed inductively as follows. Let  $\underline{W}$  be the lowest and  $\overline{W}$  be the highest competitive equilibrium of the market*

<sup>4</sup>A vector is considered less than or equal to another vector if it is less than or equal to the other vector in every component

<sup>5</sup>It is not hard to show that  $\mathbb{W}(M)$  is a closed compact set

<sup>6</sup>The inverse is not true.

*M. For an arbitrary buyer  $i$  and an arbitrary good  $j$ , let  $\bar{W}_{-i}$  be the highest competitive equilibrium of the market  $M_{-i}$  (i.e. the market without buyer  $i$ ) and let  $\underline{W}^{-j}$  be the lowest competitive equilibrium of the market  $M^{-j}$  (i.e. the market without good  $j$ ). The following inductive statements fully characterize the prices/payoffs at the lowest/highest competitive equilibrium of  $M$ :*

$$I. u_i(\underline{W}) = u_i(p(\bar{W}_{-i})).$$

$$II. p^j(\bar{W}) = p^j(u(\underline{W}^{-j})).$$

*Furthermore, the following inequalities always hold:*

$$III. p^j(\underline{W}) \leq p^j(\bar{W}_{-i}), \text{ in particular, if } j = \mu(i) \text{ then } p^j(\underline{W}) = p^j(\bar{W}_{-i}).$$

$$IV. u_i(\bar{W}) \leq u_i(\underline{W}^{-j}), \text{ in particular, if } j = \mu(i) \text{ then } u_i(\bar{W}) = u_i(\underline{W}^{-j}).$$

Note that just 2.I and 2.II are enough to fully characterize the lowest/highest competitive equilibria because we can fully specify any competitive equilibrium by specifying either the prices or the payoffs. Intuitively, we can interpret them as the following:

- (2.I) We can compute the payoff of any buyer  $i$  at the lowest competitive equilibrium of  $M$  by doing the following. Remove  $i$  from the market. Compute the prices at the highest competitive equilibrium of the rest of the market. Then, bring buyer  $i$  back to the market. The payoff that buyer  $i$  gets from her most preferred good at these prices is equal to her payoff at the lowest competitive equilibrium of the market  $M$ .
- (2.II) We can compute the price of any good  $j$  at the highest competitive equilibrium of the market  $M$  by doing the following. Remove good  $j$  from the market. Compute the buyers' payoffs at the lowest competitive equilibrium of the rest of the market. Then, bring good  $j$  back to the market. Ask each of the buyers to name a price for good  $j$  that would give them the same payoff as what they get in the lowest competitive equilibrium of the market without good  $j$ . Take the maximum among the named prices and that will be the price of good  $j$  at the highest competitive equilibrium of the whole market.

Next, we combine the above two characterization to reveal a striking similarity between the prices of the lowest competitive equilibrium and VCG payments. Note that social welfare is not even well-defined for a market with general utilities so VCG is inapplicable.

By combining the (2.I) and (2.II) we get the following interpretations for the prices of goods at the lowest competitive equilibrium. WLOG, we give the interpretation for some arbitrary good  $j$  which is allocated to buyer  $i$  at the lowest competitive equilibrium. Notice the striking similarity between these interpretation and the VCG payments “*The price that buyer  $i$  pays for good  $j$  is the lowest price at which the rest of the market becomes indifferent between buying or not buying good  $j$ . i.e., the lowest price for good  $j$  such that there is a competitive equilibrium for the market without  $i$  and  $j$  such that no buyers would strictly prefer good  $j$  to her current allocation. In other words, the price that buyer  $i$  has to pay to get good  $j$  is equal to how much good  $j$  is worth to the rest of the market.*”

**Theorem 3** (Group Strategyproofness). *A mechanism that uses the allocations/prices of the lowest competitive equilibrium is group strategyproof for buyers, meaning that there is no coalition of buyers that can collude and misreport their  $u_i^j(\cdot)$  such that all of them get strictly higher payoffs (assuming that there are no side payments).*

In the rest of this section, we give a sketch of the proof of Theorem 2. We start by defining a *Tight Alternating Path*.

**Definition 4** (Tight Alternating Path). *Given a market  $M = (I, J, \{u_i^j\})$  and a competitive equilibrium  $W$  of  $M$  with a supporting matching  $\mu(W)$ , we define a Tight Alternating Path with respect to  $W$  and  $\mu(W)$  as follows. Consider the complete bipartite graph of buyers/goods. We say the edge  $(i, j)$  is tight iff  $u_i(W) = u_i^j(p^j(W))$ . A tight alternating path is a path consisting of tight edges where every other edge on the path belongs to  $\mu(W)$ . A tight alternating path may start at either a buyer or a good and may end at either a buyer or a good. In particular, the end points of the path may be unmatched in  $\mu(W)$ . For example consider a tight alternating path  $(i_1, j_1, i_2, j_2, i_3)$  where  $i_1$  is matched with  $j_1$  and  $i_2$  is matched with  $j_2$  and  $i_3$  is unmatched in  $\mu(W)$ . In particular that means  $u_{i_3}(W) = 0 = u_{i_3}^{j_2}(p^{j_2}(W))$ .*

**Definition 5** (Demand Sets). *Given a market  $M = (I, J, \{u_i^j\})$ , we denote the demand set of a subset of buyers  $S$  at prices  $\mathbf{p}$  by  $D_S(\mathbf{p})$ . Similarly, we denote the demand set of a subset of goods  $T$  at payoffs  $\mathbf{u}$  by  $D^T(\mathbf{u})$ . Formally:*

$$D_S(\mathbf{p}) = \{j \in J \mid \exists i \in S : u_i^j(\mathbf{p}^j) = u_i(\mathbf{p})\} \quad (4.9)$$

$$D^T(\mathbf{u}) = \{i \in I \mid \exists j \in T : p_i^j(\mathbf{u}_i) = p^j(\mathbf{u})\} \quad (4.10)$$

For a competitive equilibrium  $W$  of  $M$ , we use  $D_S(W)$  to denote  $D_S(p(W))$  and  $D^T(W)$  to denote  $D^T(u(W))$ .

**Lemma 1** (Tightness). *Given a market  $M = (I, J, \{u_i^j\})$ , and a competitive equilibrium  $W$  of  $M$ :*

- *$W$  is the lowest competitive equilibrium of  $M$  iff for every subset  $T$  of the goods with strictly positive prices we have  $|D^T(W)| \geq |T| + 1$ , i.e. at least  $|T| + 1$  buyers are interested in  $T$ .*
- *$W$  is the highest competitive equilibrium of  $M$  iff for every subset  $S$  of the buyers with strictly positive payoffs we have  $|D_S(W)| \geq |S| + 1$ , i.e. the buyers in  $S$  are interested in at least  $|S| + 1$  goods.*

To see why Lemma 1 is true intuitively, assume that  $W$  is the lowest competitive equilibrium of a market  $M$  but there is a subset  $T$  of goods such that  $|D^T(W)| = |T|$ . Then, we could decrease the prices of goods in  $T$  down to the point where either a buyer out of  $D^T(W)$  becomes indifferent between her current allocation and some good in  $T$ ; or one of the goods in  $T$  hit the price of 0. But then, we get a competitive equilibrium less than  $W$  which is a contradiction. Although this lemma seems very intuitive, its formal proof turns out to be quite challenging. Most of the appendix is devoted to proving this lemma. The next lemma states a critical property of the lowest/highest competitive equilibria.

**Lemma 2** (Critical Alternating Paths). *Given a market  $M = (I, J, \{u_i^j\})$ , and a competitive equilibrium  $W$  of  $M$  with a supporting matching  $\mu$ :*

- *Iff  $W$  is the highest competitive equilibrium of  $M$  then for any good  $j$  there exists a tight alternating path from  $j$  to a buyer with a payoff of 0 or to an unmatched good. The alternating path must start with a matching edge or be of length 0.*
- *Iff  $W$  is the lowest competitive equilibrium of  $M$  then for any buyer  $i$  there exists a tight alternating path from  $i$  to a good with a price of 0 or to an unmatched buyer. The alternating path must start with a matching edge or be of length 0.*

We refer to such a tight alternating path as a “Critical Alternating Path”.

*Proof.* We only prove the first statement since the second one is similar (completely symmetric): If either  $j$  is unmatched or the payoff of buyer who is matched to  $j$  is 0 we are done. Otherwise, we run the following algorithm while maintaining a subset  $T$  of goods and a subset  $S$  of buyers with strictly positive payoffs such that there is a tight alternating path from  $j$  to each good in  $T$  and each buyer in  $S$  and  $\mu(S) = T$ . Initially, we set  $T \leftarrow \{j\}$  and  $S \leftarrow \{\mu^{-1}(j)\}$ . The repeating step is as follows: Since all buyers in  $S$  have strictly positive payoffs, we can apply Lemma 1 and argue that  $D_S(W) \geq |S| + 1 = |T| + 1$ . So, there must be a buyer  $i^*$  in  $S$  that has a tight edge to some good  $j'$  not in  $T$ . If either  $j'$  is unmatched or  $i' = \mu^{-1}(j')$  has a payoff of 0 then we are done. Otherwise, add  $j'$  to  $T$  and add  $i'$  to  $S$  and repeat. Note that we always find a critical alternating path in at most  $|I| - 1$  iterations. The “only if” direction is trivial by applying Lemma 1.  $\square$

Next, we give a sketch the proof of our main theorem. The complete proof can be found in the appendix.

*Proof sketch of Theorem 2.* We only give a sketch of the proof of (2.I) and (2.III). The proofs of (2.II) and (2.IV) are completely symmetric to the other two.

The plan of the proof is as follows. We remove an arbitrary buyer  $i$  from the market and compute the highest competitive equilibrium of the rest of the market. We then show that the prices at the highest competitive equilibrium of the market without  $i$  leads to a valid competitive equilibrium for the whole market (including buyer  $i$ ) but with a possibly different matching. We also show that the induced payoff of buyer  $i$  from these prices is the same as her payoff at the lowest competitive equilibrium of the whole market. The detail of the construction is as follows.

Choose an arbitrary buyer  $i \in I$ . Let  $M_{-i}$  denote the market without buyer  $i$  and let  $\bar{W}_{-i}$  be the highest competitive equilibrium of the market  $M_{-i}$ . Note that the market  $M_{-i}$  is of size  $|I| + |J| - 1$  so by inductively applying Theorem 2 to  $M_{-i}$  we can argue that there exists a competitive equilibrium for the market  $M_{-i}$ , so the highest competitive equilibrium of  $M_{-i}$  is well-defined. Let  $\mathbf{p} = p(\bar{W}_{-i})$  be the prices at  $\bar{W}_{-i}$ . We claim that using the prices  $\mathbf{p}$  for the market  $M$  leads to a valid competitive equilibrium  $W$ . In particular, all the prices/payoffs in  $W$  are the same as the prices/payoffs in  $\bar{W}_{-i}$  and also the payoff of buyer  $i$  is  $u_i(\mathbf{p})$  however the matching might be different. To obtain a supporting matching for  $W$ , we start with a supporting matching for  $\bar{W}_{-i}$  and modify it as follows. If  $u_i(\mathbf{p}) = 0$  then we can leave buyer  $i$  unmatched and the matching does not need to be changed. Otherwise, if  $u_i(\mathbf{p}) > 0$  then let  $j$  be the good from which buyer  $i$  achieves her highest payoff at the current prices, i.e.  $u_i(\mathbf{p}) = u_i^j(\mathbf{p}^j)$ . By applying Lemma 2 to the market  $M_{-i}$ , we can argue that there is a tight alternating path from  $j$  either to an unmatched good with a price of 0 or to a buyer with a payoff of 0. In both cases, we can match good  $j$  to buyer  $i$  and then switch the matching edges along the alternating path to get a new matching that supports  $W$ . Note that if the alternating path ends in a buyer with a payoff of 0 then the last edge of the alternating path was a matching edge and that buyer is now unmatched in the new matching, but she still has a payoff of 0. The complete proof can be found in the appendix.

Observe that from the view point of buyer  $i$ , this is a posted price mechanism with posted price vector  $p(\bar{W}_{-i})$  which does not depend on  $i$ 's reported utility (note that the choice of  $i$  was arbitrary).  $\square$

## 5 More General Models

In this section, we consider competitive equilibria in markets where both sides consist of agents with general utility function. We show a reduction from this model to the simpler model with goods and buyers. We also characterize a class of price discriminating truthful mechanisms based on these markets.

Consider the matching markets of the form  $M = (I, J, \{u_i^j\}, \{q_i^j\})$  with two sets of agents  $I$  and  $J$  such that if  $i \in I$  and  $j \in J$  are matched and  $x$  amount of money is transferred from  $i$  to  $j$ , then the utility of  $i$  is given by  $u_i^j(x)$  and the utility of  $j$  is given by  $q_i^j(-x)$  (note that  $x$  might be negative). We assume  $u_i^j$ 's and  $q_i^j$ 's have the same properties we assumed in the buyer/good model (e.g. continuous, decreasing, etc). A competitive equilibrium is also defined similarly. Despite its apparent generality, this model can be reduced to the buyer/good model as the following theorem states:

**Corollary 1.** *Given a market  $M = (I, J, \{u_i^j\}, \{q_i^j\})$  with agents on both sides, we can construct a market  $M' = (I, J, \{u_i^j\})$  with buyers/goods in which  $u_i^j(\cdot) = u_i^j(-q_i^j(-1)(\cdot))$ . Then, every competitive equilibrium in  $M'$  corresponds to a competitive equilibrium in  $M$  and vice versa with the exact same payoffs. Therefore, all of the results that we proved in the previous sections carry over to these markets. Furthermore, the mechanism that selects the lowest competitive equilibrium of  $M'$  (which is also the lowest competitive equilibrium of  $M$ ) is group strategyproof for agents of type  $I$ . Note that we could change the role of  $I$  and  $J$  and get a similar results for agents of type  $J$ . Observe that the lowest competitive equilibrium of agents of type  $J$  is the highest competitive equilibrium of agents of type  $I$  and vice versa.*

Next, we present a class of price discriminating truthful mechanisms based on the same idea:

**Corollary 2.** *Given a market  $M = (I, J, \{u_i^j\})$ , the seller(s) can personalize the price for each good/buyer by applying an arbitrary continuous and increasing function  $g_i^j(\cdot)$  to the primary price of the good  $j$ . In other words, if the price of a good  $j \in J$  at the equilibrium is  $\mathbf{p}^j$  then the price observed by agent  $i \in I$  is  $g_i^j(\mathbf{p}^j)$ . Note that  $g_i^j(\cdot)$ 's should be fixed in advance and should not depend on the reports of buyers. It is easy to see that every competitive equilibrium in this market correspond to a competitive equilibrium in the market  $M' = (I, J, \{u_i^j\})$  where  $u_i^j(\cdot) = u_i^j(g_i^j(\cdot))$  and vice versa. Consequently, all of the results that we proved in the previous sections carry over to these markets/mechanisms.*

Intuitively, the above two theorems suggest that we can write the utility functions of the agents in set  $I$  in terms of the payoffs of the agents in set  $J$  (or in terms of the primary prices in the case of personalized prices). We can then treat the agents on set  $J$  as goods and their payoffs as the prices of these goods. Note that to maintain the group strategyproofness, it is crucial that  $g_i^j(\cdot)$ 's be fixed in advance and not depend on the reports of buyers. In the next section, we present a practical application of this idea.

## 6 Application to Ad-Auctions

In this section, we present a truthful mechanism for Ad-auctions that combines pay per click (a.k.a charge per click or CPC) and pay per impression (a.k.a CPM) advertisers with general utility functions. In particular, our mechanism is group strategyproof regardless of whether the search engine uses the correct clickthrough rates or whether advertisers agree with the search engine on clickthrough rates.

We formally define our model as follows. Given a set of advertisers  $I$  and a set of slots  $J$ , we assume that utility of advertiser  $i$  from slot  $j$  is given by  $u_i^j(x)$ <sup>7</sup> where  $x$  is payment per click for CPC advertisers and payment per impression for CPM advertisers. We say that a CPC advertiser  $i$  has standard utility function if for all slots  $j$ :  $u_i^j(x) = c_i^j(v_i^j - x)$  in which  $v_i^j$  is the advertiser's value for a click on slot  $j$  and  $c_i^j$  is the advertiser's belief about her clickthrough rate (CTR). We also say that a CPM advertiser  $i$  has standard utility function if for all slots  $j$ :  $u_i^j(x) = v_i^j - x$  in which  $v_i^j$  is the advertiser's value for a click on slot  $j$ . Furthermore, we assume that search engine believes that the CTR of advertiser  $i$  on slot  $j$  is  $\hat{c}_i^j$  which might be different from  $c_i^j$  (i.e., advertisers and search engine could disagree). Furthermore, we assume that  $v_i^j$  and  $c_i^j$  are advertiser's private information but  $\hat{c}_i^j$  is publicly announced.

Before we explain our mechanism, let us consider what happens if we applied VCG to this setting, assuming that we only have CPC advertisers with standard utility function. We get the following LPs. The primal computes the social welfare maximizing allocation while the dual computes prices/payoffs:

$$\begin{aligned}
\textbf{Primal:} \quad & \max \cdot \sum_{i \in I} \sum_{j \in J} c_i^j v_i^j \mathbf{x}_i^j \\
& \forall i \in I : \quad \sum_{j \in J} \mathbf{x}_i^j \leq 1 \\
& \forall j \in J : \quad \sum_{i \in I} \mathbf{x}_i^j \leq 1 \\
& \quad \quad \quad \mathbf{x}_i^j \geq 0 \\
\textbf{Dual:} \quad & \min \cdot \sum_{i \in I} \mathbf{u}_i + \sum_{j \in J} \mathbf{p}^j \\
& \forall i \in I, \forall j \in J : \quad \mathbf{u}_i + \mathbf{p}^j \geq c_i^j v_i^j \\
& \quad \quad \quad \mathbf{u}_i \geq 0 \\
& \quad \quad \quad \mathbf{p}^j \geq 0
\end{aligned} \tag{6.1}$$

The set of solutions to the dual program would be the set of competitive equilibria of the market and the one with the lowest prices would correspond to the lowest competitive equilibrium which would also coincide with the VCG payments/payoffs. However, the problem is that payments should be charged *per click* while  $\mathbf{p}^j$  represents the expected payment, i.e., payment *per impression*. So, per click payments are given by  $\mathbf{p}^j / \hat{c}_i^j$ . However, by dividing by  $\hat{c}_i^j$ , we lose the strategyproofness guarantee if  $c_i^j$  and  $\hat{c}_i^j$  are not the same (i.e., if advertisers and search engine have different estimates about clickthrough rates). Note that we cannot use  $\mathbf{p}^j / c_i^j$  either because then advertisers have the incentive to untruthfully report a higher  $c_i^j$  which would give them a higher chance of winning a better slot and at the same time would lower their payment. Next, we present a mechanism that also addresses this problem.

**Mechanism 1.** Compute the lowest competitive equilibrium of the market  $M = (I, J, \{u_i^j\})$  using the following personalized prices by applying Corollary 2. For each advertiser  $i$  and slot  $j$  we define a personalized price  $g_i^j(\cdot)$  as follows. If  $i$  is a CPC advertiser then we define  $g_i^j(x) = x / \hat{c}_i^j$  in which  $\hat{c}_i^j$  is the estimate of the search engine for the clickthrough rate of advertiser  $i$  on slot  $j$ . If  $i$  is a CPM advertiser then we define  $g_i^j(x) = x$ .

**Remark 1** (Interpretation of mechanism 1). We can conceptually reinterpret this mechanism as an ascending auction as follows. Initially, we assign a primary price of 0 to every slot and during the auction whenever the demand for a slot is more than one, we increase the primary price of that slot. At any time during the auction, each advertiser demands one of the slots at the current prices. However, different advertisers see different prices. At any point during the auction, advertiser  $i$  observes a price of  $g_i^j(\mathbf{p}^j)$  for slot  $j$  in which  $\mathbf{p}^j$  is the primary price of the slot  $j$ . The auction stops when there is no over demanded slot. Intuitively,  $\mathbf{p}^j$  denotes the expected revenue of the

<sup>7</sup> $u_i^j(x)$  must be continuous and decreasing in  $x$  and for a high enough  $x$  it should become non-positive

search engine from slot  $j$  and  $g_i^j(\mathbf{p}^j)$  is the price that advertiser  $i$  has to pay for each click so that the search engine makes  $\mathbf{p}^j$  in expectation.

Next theorem summarizes the important properties of the above mechanism.

**Theorem 4.** *Mechanism 1 is group strategyproof and also maximizes the social welfare in the following sense. Let  $A$  be a group of advertisers with standard utility functions who also agree with the search engine on the CTRs and let  $A'$  be the rest of the advertisers. Let  $s$  denote the search engine. Then mechanism 1 maximizes the welfare of  $\{s\} \cup A$  and also the presence of  $A'$  may not decrease the welfare of  $\{s\} \cup A$ . In particular, if all advertisers have standard utility functions and agree with the search engine on the CTRs then the outcome of this mechanism coincides exactly with the VCG outcome.*

Notice that mechanism 1 is group strategyproof regardless of whether the search engine and advertisers have the same estimates about the clickthrough rates. This answers an open question raised by [1]. As for existing CPC ad-auction mechanisms, that we are aware of, incentive compatibility relies on everyone agreeing on clickthrough rate estimates made by the search engine.

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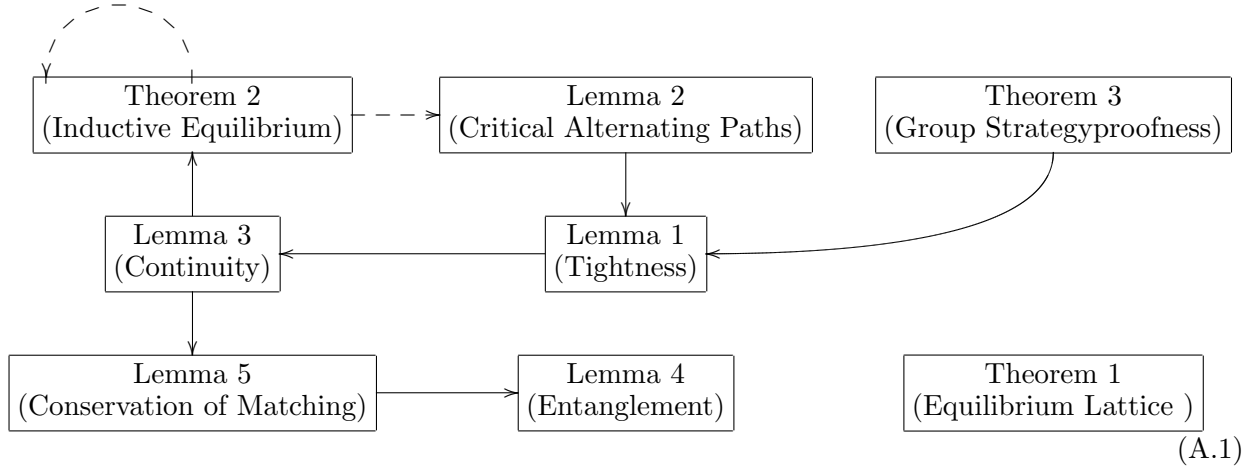
## References

- [1] G. Aggarwal, S. Muthukrishnan, D. Pál, and M. Pál. General auction mechanism for search advertising. In *Proceedings of the 18th international conference on World wide web, WWW '09*, pages 241–250, New York, NY, USA, 2009. ACM.
- [2] I. Ashlagi, M. Braverman, A. Hassidim, R. Lavi, and M. Tennenholtz. Position auctions with budgets: Existence and uniqueness. *The B.E. Journal of Theoretical Economics*, 10(1), 2010.
- [3] L. M. Ausubel and P. Milgrom. Ascending auctions with package bidding. Working Papers 02004, Stanford University, Department of Economics, Mar. 2002.
- [4] G. Demange and D. Gale. The strategy structure of two-sided matching markets. *Econometrica*, 53(4):873–88, July 1985.
- [5] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–72, 1986.
- [6] D. Gale. Equilibrium in a discrete exchange economy with money. *International Journal of Game Theory*, 13(1):61–64, 1984.
- [7] D. Gale and L. S. Shapley. College admissions and the stability of marriage. volume 69, pages 9–15, January 1962.
- [8] F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, July 1999.

- [9] J. W. Hatfield and S. D. Kominers. Matching in networks with bilateral contracts: extended abstract. In *Proceedings of the 11th ACM conference on Electronic commerce*, EC '10, 2010.
- [10] J. W. Hatfield and P. R. Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, September 2005.
- [11] J. Kelso, Alexander S and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, November 1982.
- [12] B. Knaster, C. Kuratowski, and S. Mazurkiewicz. Ein beweis des fixpunktsatzes für  $n$ -dimensionale simplexe. *Fund. Math*, 14:132–137, 1929.
- [13] H. W. Kuhn. Variants of the hungarian method for assignment problems.
- [14] C. E. Lemke and J. Howson, J. T. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):413–423, 1964.
- [15] H. B. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, 91(3):461–79, 1983.
- [16] S. Mamer, J. and Bikhchandani. *Journal of Economic Theory*, 74:385–413, 1997.
- [17] M. Quinzii. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory*, 13:41–60, 1984.
- [18] H. E. Scarf. The core of an  $n$  person game. *Econometrica*, 35(1), January 1967.
- [19] L. Shapley and M. Shubik. The assignment game i: The core. 1(1):111–130, 1971.

## A Other Results & Proofs

In this section, we present the missing proofs and several other results. Before we proceed, we should mention that some of our theorems/lemmas mutually depend on each other. However, this does not create a cycle in the proofs. The following diagram illustrates the dependencies in the proofs. A solid arrow from  $A$  to  $B$  means that when  $A$  is invoked on a market of size  $n$ , the proof of  $A$  invokes  $B$  on a market of the same size. A dashed arrow from  $A$  to  $B$  means that  $A$  invokes  $B$  on a market of strictly smaller size. We can then prove all of the lemmas/theorems by induction on the size of the market. i.e. we assume that all of the lemmas/theorem are true for markets of size less than  $n$  and then all of the lemmas/theorems can be proved for markets of size  $n$ .



Note that Theorem 1 is implicitly used by most of the other lemmas/theorems so we didn't display the dependencies on it in the above diagram.

We proceed by defining *Bounded Competitive Equilibrium*:

**Definition 6** ( $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -Bounded Competitive Equilibrium). *Given a market  $M = (I, J, \{u_i^j\})$  and a lower bound price vector  $\underline{\mathbf{p}} \geq \mathbf{0}$  and a lower bound payoff vector  $\underline{\mathbf{u}} \geq \mathbf{0}$ , we say that  $W$  is a  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M$  iff  $W$  is a competitive equilibrium of  $M$  and  $p(W) \geq \underline{\mathbf{p}}$  and  $u(W) \geq \underline{\mathbf{u}}$ .*

*Note that for a given  $\underline{\mathbf{u}}$  and  $\underline{\mathbf{p}}$ , the  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibria of  $M$  form a complete lattice which is a complete sublattice of all of the competitive equilibria of  $M$ . In particular, there is a lowest and a highest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M$ . Notice that for arbitrary  $\underline{\mathbf{u}}$  and  $\underline{\mathbf{p}}$ , a  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium does not necessarily exist.*

The next lemma provides the basic ingredient for the proof of Lemma 1.

**Lemma 3** (Continuity). *Assume a market  $M = (I, J, \{u_i^j\})$  with  $|I| = |J|$  and lower bounds  $\underline{\mathbf{p}} \geq \mathbf{0}$  and  $\underline{\mathbf{u}} \geq \mathbf{0}$  on the prices/payoffs, such that a  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium exists. Then, at the lowest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium, there exists at least one good  $j^* \in J$  whose price is exactly equal to its lower bound (i.e.  $\mathbf{p}^{j^*}$ ). Similarly, at the highest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium, there exists at least one buyer  $i^* \in I$  whose payoff is exactly equal to her lower bound (i.e.  $\underline{\mathbf{u}}_{i^*}$ ).*

To give more intuition on the above statements, consider the following immediate corollary which we can derive by setting no lower bounds (i.e. a lower bound of  $\mathbf{0}$ ) on the prices/payoff:

**Corollary 3.** *Given a market  $M = (I, J, \{u_i^j\})$  with  $|I| = |J|$  the following statements are always true:*

- *At the lowest competitive equilibrium, there is at least one good that has a price of 0.*
- *At the highest competitive equilibrium there is at least one buyer that achieves a payoff of 0.*

As another immediate corollary, Lemma 3 shows that there is a continuum of equilibria between the lowest and the highest competitive equilibria:

**Corollary 4.** *Given a market  $M = (I, J, \{u_i^j\})$  with  $|I| = |J|$ , with  $\underline{W}$  and  $\overline{W}$  being the lowest and the highest competitive equilibria respectively, there is a continuum of equilibria between  $\underline{W}$  and  $\overline{W}$ .*

*Proof.* Define  $p(t) = (1 - t)p(\underline{W}) + tp(\overline{W})$ . Now by applying Lemma 3 we can get a continuum of equilibria by computing the lowest  $(0, p(t))$ -bounded competitive equilibrium for each  $t \in [0, 1]$ .  $\square$

Before we can proceed further, we need two more lemmas. The next two lemmas show very basic properties of competitive equilibria in unit-demand markets:

**Lemma 4** (Entanglement). *Given a market  $M = (I, J, \{u_i^j\})$  and a competitive equilibria  $W$  of  $M$ . If buyer  $i$  is matched with good  $j$  at  $W$  then the price of good  $j$  and the payoff of buyer  $i$  are entangled in any other competitive equilibrium of  $M$  which means at any other competitive equilibrium like  $W'$  if the price of good  $j$  is higher then the payoff of buyer  $i$  must be lower and vice versa. Note that this claim is true regardless of whether buyer  $i$  and good  $j$  are actually matched to each other in  $W'$ .*

*Proof.* Let  $W'$  be any other competitive equilibrium of  $M$ . Partition the buyers to  $S$ ,  $S'$  and  $S''$  such that buyers in  $S$  have a higher payoff at  $W$ , buyers in  $S'$  have a higher payoff at  $W'$  and buyers in  $S''$  have the same payoff at both  $W$  and  $W'$ . Similarly, partitions the goods to  $T$ ,  $T'$  and  $T''$  such that goods in  $T$  have a higher price at  $W$ , goods in  $T'$  have a higher price at  $W'$  and goods in  $T''$  have the same price at both  $W$  and  $W'$ . It is easy to show the following statements are true using the definition of competitive equilibria and the fact that both  $W$  and  $W'$  are competitive equilibria:

- At  $W$ , all buyers in  $S$  must be matched to goods in  $T'$  so  $|S| \leq |T'|$ .
- At  $W'$ , all goods in  $T'$  must be matched to buyers in  $S$  so  $|T'| \leq |S|$ .

From the above statement, we can conclude  $|S| = |T'|$  and buyers in  $S$  and goods in  $T'$  must be matched to each other at both equilibria. Similarly:

- At  $W$ , all goods in  $T$  must be matched to buyers in  $S'$  so  $|T| \leq |S'|$ .
- At  $W'$ , all buyers in  $S'$  must be matched to goods in  $T$  so  $|S'| \leq |T|$ .

So, we can conclude  $|S'| = |T|$  and buyers in  $S'$  and goods in  $T$  must be matched to each other at both equilibria. Furthermore, we can then conclude that buyers in  $S''$  and goods in  $T''$  may only be matched to each other. That proves the claim of the lemma.  $\square$

**Lemma 5** (Conservation of Matching). *Given a market  $M = (I, J, \{u_i^j\})$ , for any  $i \in I$ , if there exists a competitive equilibrium  $W$  of  $M$  at which buyer  $i$  has a strictly positive payoff (i.e.  $u_i(W) > 0$ ) then buyer  $i$  is never unmatched in any competitive equilibrium of  $M$ . Similarly, for any  $j \in J$ , if there exists a competitive equilibrium  $W$  of  $M$  at which good  $j$  has a strictly positive price (i.e.  $p^j(W) > 0$ ) then good  $j$  is never unmatched at any competitive equilibrium of  $M$ .*

*Proof.* Let  $W'$  be any other competitive equilibrium of the market. Partition the buyers to  $S$ ,  $S'$ ,  $S''$  and partition the goods to  $T$ ,  $T'$ ,  $T''$  as in the proof of Lemma 4. In the proof of that lemma, we showed that the matching pairs can only be from  $S$  and  $T'$  or  $S'$  and  $T$  or  $S''$  and  $T''$ . Let  $i$  be any buyer with strictly positive payoff at  $W$ , if  $i$  has strictly positive payoff at  $W'$  then it must also be matched at  $W'$ . Otherwise if the payoff of buyer  $i$  is 0 at  $W'$  then  $i$  must belong to set  $S$ . We know that  $|S| = |T'|$  and that all the goods in  $T'$  must have strictly positive prices so they must all be matched and therefore all buyers in  $S$ , including buyer  $i$ , must also be matched. The second statement can be proved by a similar argument for good  $j$ .  $\square$

*Proof of Lemma 3.* We only prove the first claim. The proof of the second claim is similar (completely symmetric). The plan of the proof is as follows:

First, we define a transformed market  $M' = (I, J, \{u_i^j\})$  with  $u_i^j(x) = u_i^j(x + \mathbf{p}^j) - \underline{\mathbf{u}}_i$ . We claim that there is a one-to-one mapping between  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibria of the original market and the competitive equilibria of the transformed market. Formally, a  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium  $W$  of  $M$  corresponds to a competitive equilibrium  $W'$  of  $M'$  such that  $p(W') = p(W) - \underline{\mathbf{p}}$  and  $u(W') = u(W) - \underline{\mathbf{u}}$  and  $\mu(W') = \mu(W)$ . We then show that there is competitive equilibrium of  $M'$  in which there is a good with a price of 0 which then means in the corresponding competitive equilibrium of the original market the price of that good is equal to its lower bound and therefore at the lowest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M$  the price of that good must also be equal to its lower bound which proves the claim.

We now prove that there is a good with a price of 0 at the lowest competitive equilibrium of  $M'$ . We choose an arbitrary buyer  $i$  from  $M'$  and remove it from the market. Let  $\bar{W}_{-i}$  be the highest competitive equilibrium of the remaining market. By the assumption of the lemma, we know  $|I| = |J|$  and so in  $\bar{W}_{-i}$  there are more goods than there are buyers so there must be an unmatched good which we denote by  $j^*$ . Note that the price of  $j^*$  in  $\bar{W}_{-i}$  must be 0. On the other hand, by applying Theorem 2 to  $M'$  and using (2.III) we have  $p^j(W) \leq p^j(\bar{W}_{-i})$  for every good  $j$ . Therefore, it must be that the price of  $j^*$  in  $W$  is also 0 and that completes the proof.

There is a subtlety that we should point out about the one-to-one mapping between the competitive equilibria of the original market and those of the transformed market. It is clear that every  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M$  can be transformed to a competitive equilibrium of  $M'$ . However, for the other direction, we need to show that all goods/buyers are matched, otherwise after applying the inverse transform we may end up with an unmatched good/buyer that has a positive price/payoff. To show that all buyers/goods are matched in every competitive equilibrium of  $M'$ , we can apply Lemma 5. To apply that lemma, we only need to show that there is a competitive equilibrium of  $M'$  in which all goods have strictly positive prices and then by that lemma all the goods must always be matched (and so do all buyers because  $|I| = |J|$ ). Notice that if there is no competitive equilibrium for  $M'$  in which all goods have strictly positive prices then either in every  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M$  there is a good whose price is equal to its lower bound or  $M$  has no  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium at all which either way trivially proves the claim of this lemma.  $\square$

*Proof of Lemma 1.* We only prove the first statement. The proof of the second statement is similar (completely symmetric).

First, we prove the “only if” direction. Assume that  $W$  is the lowest competitive equilibrium of  $M$ . For every subset  $T$  of goods with strictly positive prices, we prove that  $D^T(W) \geq |T| + 1$ , i.e. there are at least  $|T| + 1$  buyers who are interested in some good in  $T$ . The proof is as follows. Since all the goods in  $T$  have strictly positive prices, they must all be matched. Let  $S$  be the subset of buyers that are matched to  $T$ . Notice that  $S \subset D^T(W)$  and  $|S| = |T|$ . So, to complete the proof we only need to show that there is one more buyer not in  $S$  who is also interested in a good in  $T$ . Let  $\underline{\mathbf{p}}$  be the prices induced by payoffs of buyers not in  $S$ , i.e.  $\underline{\mathbf{p}}^j = \max_{i \in I-S} p_i^j(u_i(W))$ . Similarly, let  $\underline{\mathbf{u}}$  be the payoffs induced by the prices of goods not in  $T$ , i.e.  $\underline{\mathbf{u}}_i = \max_{j \in J-T} u_i^j(p^j(W))$ . Notice that  $W$  is a  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibria of the market  $M' = (S, T, \{u_i^j\})$ . Furthermore, if we replaced the part of  $W$  that corresponds to  $S$  and  $T$  with any other  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M'$ , we would get a valid competitive equilibrium for  $M$  which implies that  $W$  must be the lowest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M'$  as well because otherwise we could replace the part of  $W$  corresponding to  $S$  and  $T$  with the lowest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M'$  and get a lower competitive equilibrium for  $M$  which would contradict  $W$  being

the lowest competitive equilibrium of  $M$ . Because  $W$  is the lowest  $(\underline{\mathbf{u}}, \underline{\mathbf{p}})$ -bounded competitive equilibrium of  $M'$ , by applying Lemma 3 to the market  $M'$ , we can argue that there is a good  $j^* \in T$  such that  $p^{j^*}(W) = \underline{\mathbf{p}}^{j^*}$ . Since all the goods in  $T$ , including  $j^*$ , have strictly positive prices,  $\underline{\mathbf{p}}^{j^*}$  must also be strictly positive and because of the way we defined  $\underline{\mathbf{p}}$  there must be a buyer  $i^*$  not in set  $S$  such that  $\underline{\mathbf{p}}^{j^*} = p_{i^*}^{j^*}(\mathbf{u}_{i^*})$ . That means  $i^*$  must be interested in good  $j^*$  and therefore  $\{i^*\} \cup S \subset D^T(W)$  which proves that there are at least  $|T| + 1$  buyers interested in the goods in  $T$ .

The proof of the “if” direction is trivial. The proof is by contradiction. Let  $W$  be a competitive equilibrium of  $M$  such that for every subset  $T$  of goods with strictly positive prices we have  $D^T(W) \geq |T| + 1$ . Let  $\underline{W}$  be the lowest competitive equilibrium of  $M$  and assume that  $W$  and  $\underline{W}$  are not the same. Let  $T$  consist of all the goods that have a higher price at  $W$  compared to  $\underline{W}$ . We know that there are at least  $|T| + 1$  buyers interested in  $T$  at  $W$  and these buyers must have higher payoffs at  $\underline{W}$  because the prices of the goods in  $T$  are strictly lower. Therefore, the goods assigned to these buyers at  $\underline{W}$  must have lower prices and so there are at least  $|T| + 1$  goods that have higher prices at  $W$  compared to  $\underline{W}$  which contradicts the assumption that  $T$  was the set of all the goods that had higher prices at  $W$ .  $\square$

Next, we present the complete proof of Theorem 2:

*Proof of Theorem 2.* We only prove (2.I) and (2.III). The proofs of (2.II) and (2.IV) are completely symmetric to the other two.

The plan of the proof is as follows. We remove an arbitrary buyer  $i$  from the market and compute the highest competitive equilibrium of the rest of the market. We then show that the prices at the highest competitive equilibrium of the market without  $i$  leads to a valid competitive equilibrium for the whole market (including buyer  $i$ ) but with a possibly different matching. We also show that the induced payoff of buyer  $i$  from these prices is the same as her payoff at the lowest competitive equilibrium of the whole market. The detail of the construction is as follows.

Choose an arbitrary buyer  $i \in I$ . Let  $M_{-i}$  denote the market without buyer  $i$  and let  $\overline{W}_{-i}$  be the highest competitive equilibrium of the market  $M_{-i}$ . Note that the market  $M_{-i}$  is of size  $|I| + |J| - 1$  so by inductively applying Theorem 2 to  $M_{-i}$  we can argue that there exists a competitive equilibrium for the market  $M_{-i}$ , so the highest competitive equilibrium of  $M_{-i}$  is well-defined. Let  $\mathbf{p} = p(\overline{W}_{-i})$  be the prices at  $\overline{W}_{-i}$ . We claim that using the prices  $\mathbf{p}$  for the market  $M$  leads to a valid competitive equilibrium  $W$ . In particular, all the prices/payoffs at  $W$  are the same as the prices/payoffs at  $\overline{W}_{-i}$  and also the payoff of buyer  $i$  is  $u_i(\mathbf{p})$ , however the matching might be different. To obtain a supporting matching for  $W$ , we start with a supporting matching for  $\overline{W}_{-i}$  and modify it as follows. If  $u_i(\mathbf{p}) = 0$  then we can leave buyer  $i$  unmatched and the matching does not need to be changed. Otherwise, if  $u_i(\mathbf{p}) > 0$  then let  $j$  be the good from which buyer  $i$  achieves her highest payoff at the current prices, i.e.  $u_i(\mathbf{p}) = u_i^j(\mathbf{p}^j)$ . By applying Lemma 2 to the market  $M_{-i}$ , we can argue that there is a tight alternating path from  $j$  either to an unmatched good with a price of 0 or to a buyer with a payoff of 0. In both cases, we can match good  $j$  to buyer  $i$  and then switch the matching edges along the alternating path to get a new matching that supports  $W$ . Note that if the alternating path ends in a buyer with a payoff of 0 then the last edge of the alternating path was a matching edge and that buyer is now unmatched in the new matching, but she still has a payoff of 0.

Next, we prove each one of our claims:

- Proof of **Existence**: By induction, we assumed that  $M_{-i}$  must have a competitive equilibrium. We then took the highest competitive equilibrium of  $M_{-i}$  and constructed a competitive equilibrium for  $M$ . So  $M$  has a competitive equilibrium.

- Proof of (2.III)  $\boxed{p^j(\underline{W}) \leq p^j(\overline{W}_{-i})}$ : Notice that we constructed a competitive equilibrium  $W$  of the market  $M$  which has the same prices as the  $\overline{W}_{-i}$ . The prices at the lowest competitive equilibrium of  $M$  are no more than the prices at  $W$  so  $p(\underline{W}) \leq p(W) = p(\overline{W}_{-i})$ .
- Proof of (2.III)  $\boxed{p^j(\underline{W}) = p^j(\overline{W}_{-i}) \text{ when } \mu(i) = j}$ : Notice that since  $i$  and  $j$  are matched, if we remove both of them the rest of  $\underline{W}$  is still a valid competitive equilibrium for  $M_{-i}^{-j}$ . Let  $\underline{W}_{-i}^{-j}$  denote the lowest competitive equilibrium of  $M_{-i}^{-j}$ . Note that both  $\underline{W}$  and  $\underline{W}_{-i}^{-j}$  are valid competitive equilibria for  $M_{-i}^{-j}$  but  $\underline{W}_{-i}^{-j}$  is the lowest, so the prices of goods  $J - \{j\}$  might only be lower at  $\underline{W}_{-i}^{-j}$  and so the payoffs of buyers  $I - \{i\}$  might only be higher at  $\underline{W}_{-i}^{-j}$  and so the price induced by buyers  $I - \{i\}$  on good  $j$  might only be lower at  $\underline{W}_{-i}^{-j}$  than the price induced by them on good  $j$  at  $\underline{W}$ . However, by applying Theorem 2 inductively on market  $M_{-i}$  and using (2.II), we get that  $p^j(\overline{W}_{-i})$  is exactly the induced price of buyers  $I - \{i\}$  on good  $j$  at  $\underline{W}_{-i}^{-j}$ . Therefore,  $p^j(\overline{W}_{-i})$  must be less than or equal to the induced price on good  $j$  at  $\underline{W}$  which is itself less than or equal to  $p^j(\underline{W})$ . On the other hand, from the previous paragraph we have  $p^j(\underline{W}) \leq p^j(\overline{W}_{-i})$ , so the two must be equal.
- Proof of (2.I)  $\boxed{u_i(\underline{W}) = u_i(p(\overline{W}_{-i}))}$ : If  $i$  is matched with  $j$  in  $\underline{W}$  then  $u_i = u_i^j(p^j(\underline{W}))$  and by the previous statement  $p^j(\underline{W}) = p^j(\overline{W}_{-i})$ . Therefore  $u_i(\underline{W}) = u_i^j(p^j(\overline{W}_{-i})) = u_i(p(\overline{W}_{-i}))$ . The last equality follows from the fact that we chose  $j$  to be the good from which buyer  $i$  obtains her highest payoff at prices  $p(\overline{W}_{-i})$ .

□

*Proof of Theorem 3.* To prove that the mechanism that uses the lowest competitive equilibrium for allocations/payments is group strategyproof for buyers, we must show that there is no coalition of buyers who can collude such that all of them achieve strictly higher payoffs (without making side payments). The proof is by contradiction. Let  $S$  be the largest subset of buyers who can collude and possibly misreport their  $u_i^j$ 's and all of them achieve strictly higher payoffs. Let  $\underline{W}$  be the lowest competitive equilibrium of  $M$  with respect to the true utility functions and let  $W'$  be the lowest competitive equilibrium with respect to the reported utility functions assuming that buyers in  $S$  have colluded. Let  $T$  be the subset of the goods that are matched to  $S$  at  $W'$ . Since all the buyers in  $S$  are achieving strictly higher payoffs at  $W'$ , they cannot be unmatched at  $W'$  (i.e.  $|T| = |S|$ ) and the prices of the goods in  $T$  should be strictly lower at  $W'$ . That means the goods in  $T$  must have had strictly positive prices in  $\underline{W}$ . By applying Lemma 1, we argue that there must have been a subset  $S'$  of buyers of size at least  $|T| + 1$  who were interested in some good in  $T$  at  $\underline{W}$ . Observe that all of the buyers in  $S'$  must be getting a strictly higher payoff at  $W'$  because the prices of all the goods in  $T$  are strictly lower. But  $S'$  is larger than  $S$  which contradicts our assumption that  $S$  was the largest set of buyers who could all benefit from collusion. □

*Proof of Theorem 4.* The group strategyproofness follows from Theorem 3. So we only prove the second part: Assuming the mechanism has computed a lowest competitive equilibrium  $W$  as the outcome with price vector  $\mathbf{p}$ , the expected utility of advertiser  $i$  from slot  $j$  is given by  $u_i^j(g_i^j(\mathbf{p}^j))$  where  $\mathbf{p}^j$  is the base price of good  $j$  and  $g_i^j(x) = x/\hat{c}_i^j$  is the personalized price of slot  $j$  for advertiser  $i$ . So for each advertiser  $i \in A$  we have  $u_i^j(\mathbf{p}^j) = c_i^j(v_i^j - \mathbf{p}^j/\hat{c}_i^j)$ . Furthermore, since  $\hat{c}_i^j = \tilde{c}_i^j$ , we can simplify the utility function and get  $u_i^j(\mathbf{p}^j) = c_i^j v_i^j - \mathbf{p}^j$ . Now, consider the complete bipartite graph  $G$  with advertisers and slots. Let the weight of each edge  $(i, j)$  be  $c_i^j v_i^j$ . Note that for each advertiser  $i \in A$  we have  $u_i(W) + p^j(W) \geq c_i^j v_i^j$  in which  $W$  is the outcome of the mechanism.

Therefore, the total expected welfare of the coalition  $\{s\} \cup A$  is at least as much as the weight of the maximum weight matching in the absence of  $A'$ . Furthermore, if  $A'$  is empty (i.e. everyone agrees on the CTRs), the mechanism computes the efficient allocation (i.e., a maximum weight matching) and the outcome is the same as the VCG outcome.  $\square$

Next, we present the proof of the lattice structure. The proof does not use any other lemma.

*Proof of Theorem 1.* To simplify the proof, we add  $|J|$  dummy buyers and  $|I|$  dummy goods so as to make sure that we can always get a perfect matching. We set  $u_i^j(x) = -x$  whenever either  $i$  or  $j$  or both are dummy. By doing this we can always make sure that for every competitive equilibrium  $W$  there is a perfect matching  $\mu$  that supports the equilibrium. Note that (4.2) and (4.3) ensure that for any unmatched buyer  $i$ ,  $\mathbf{u}_i = 0$  and for any unmatched good  $j$ ,  $\mathbf{p}^j = 0$  so we can arbitrarily match the unmatched buyers/goods to the new dummy buyers/goods and then match the remaining dummy buyers/goods together. Observe that by adding dummy buyers/goods we don't need to be concerned about (4.2) and (4.3) anymore <sup>8</sup>.

- First, we prove that  $W_{\inf}$  is a valid competitive equilibrium :

- We first show that  $\mu_{\inf}$  is a valid matching. The proof is by contradiction. Suppose it is not. Then, there should be  $i, i' \in I$  such that  $\mu_{\inf}(i) = \mu_{\inf}(i') = j$ . Since both  $\mu$  and  $\mu'$  are valid matchings,  $j$  should be matched to  $i$  in one of them and to  $i'$  in the other one. WLOG, assume that  $\mu(i) = j$  and  $\mu'(i') = j$ . From the definition of  $\mu_{\inf}$  and because  $\mu_{\inf}(i) = \mu(i)$ , we can argue  $\mathbf{u}_i \geq \mathbf{u}'_i$ . So we have  $u_i^j(\mathbf{p}^j) = \mathbf{u}_i \geq \mathbf{u}'_i \geq u_{i'}^j(\mathbf{p}^j)$  which means  $\mathbf{p}^j \leq \mathbf{p}'^j$ . On the other hand, by repeating the same argument for  $i'$  instead of  $i$ , we can conclude that  $\mathbf{u}_{i'} < \mathbf{u}'_{i'}$  (note that according to the definition of  $\mu_{\inf}$ ,  $\mu_{\inf}(i') = \mu'(i')$  if  $\mathbf{u}_{i'} < \mathbf{u}'_{i'}$ ) and so we have  $u_{i'}^j(\mathbf{p}^j) \leq \mathbf{u}_{i'} < \mathbf{u}'_{i'} = u_{i'}^j(\mathbf{p}'^j)$  which means  $\mathbf{p}^j > \mathbf{p}'^j$ . We have a contradiction because we just proved  $\mathbf{p}^j \leq \mathbf{p}'^j$  and  $\mathbf{p}^j > \mathbf{p}'^j$ . Therefore,  $\mu_{\inf}$  must be a valid matching.
- We show that  $W_{\inf}$  satisfies the (4.1). WLOG, assume  $\mu_{\inf}(i) = \mu(i) = j$ . So  $u_i^j(\mathbf{p}^j) = \mathbf{u}_i \geq \mathbf{u}'_i \geq u_i^j(\mathbf{p}'^j)$  which means  $\mathbf{p}^j \leq \mathbf{p}'^j$ . Therefore,  $p^j(W_{\inf}) = \mathbf{p}^j$ . Together with the fact that  $u_i(W_{\inf}) = \mathbf{u}_i$  and  $\mu_{\inf}(i) = \mu(i)$  we can argue that  $W_{\inf}$  satisfies the first part of (4.1) because  $W$  satisfies it. Similarly, for any  $j'$ ,  $\mathbf{u}_i \geq u_i^{j'}(\mathbf{p}'^{j'})$  and  $\mathbf{u}'_i \geq u_i^{j'}(\mathbf{p}^{j'})$ . Therefore,  $u_i(W_{\inf}) \geq \max(u_i^{j'}(\mathbf{p}^{j'}), u_i^{j'}(\mathbf{p}'^{j'})) = u_i^{j'}(\min(\mathbf{p}^{j'}, \mathbf{p}'^{j'})) = u_i^{j'}(p^{j'}(W_{\inf}))$  so the second part of (4.1) is also satisfied.

- Next, we prove that  $W_{\sup}$  is a valid competitive equilibrium :

- We first show that  $\mu_{\sup}$  is a valid matching. The proof is by contradiction. Suppose it is not. Then, there should be  $i, i' \in I$  such that  $\mu_{\sup}(i) = \mu_{\sup}(i') = j$ . Since  $|I| = |J|$  and there are two buyers that are matched to the same good, there should be another good  $j'$  to which no buyer is matched. On the other hand, both  $\mu$  and  $\mu'$  are valid perfect matchings so  $j'$  must be matched in both of them. Let  $s = \mu^{-1}(j')$  and  $s' = \mu'^{-1}(j')$ . Notice that  $s \neq s'$  otherwise  $\mu_{\sup}(s)$  would be  $j'$  as well. Because  $j'$  is not matched in  $\mu_{\sup}$ , it should be that  $\mu_{\sup}(s) \neq j'$  which means  $\mu_{\inf}(s) = j'$ . Similarly, it should be that  $\mu_{\sup}(s') \neq j'$  which means  $\mu_{\inf}(s') = j'$ . However, that means  $\mu_{\inf}(s) = \mu_{\inf}(s') = j'$  which means  $\mu_{\inf}$  is not a valid matching which is a contradiction since we already proved  $\mu_{\inf}$  is a valid perfect matching.

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<sup>8</sup>Remember that for non-dummy buyers,  $u_i^j(0)$  might be negative so we may not be able to match the 0 priced items and 0 payoff buyers. This is a technicality that arises when we transform the market in the proof of Lemma 3

- We show that  $W_{\text{sup}}$  satisfies the (4.1). WLOG, assume  $\mu_{\text{sup}}(i) = \mu(i) = j$ . Let  $i' = \mu_{\text{inf}}^{-1}(j)$ . It must be that  $\mu'(i') = j$  (because we already know that  $\mu(i) = j$  so  $\mu(i') \neq j$ , but  $i'$  must be matched to  $j$  in  $\mu$  or  $\mu'$ ). That means  $u_{i'}^j(\mathbf{p}^j) = \mathbf{u}_{i'}' \geq \mathbf{u}_{i'} \geq u_{i'}^j(\mathbf{p}^j)$ , so  $\mathbf{p}^{j'} \leq \mathbf{p}^j$  and therefore  $p^j(W_{\text{sup}}) = \mathbf{p}^j$ . We also have  $u_i(W_{\text{sup}}) = \mathbf{u}_i$  and  $\mu_{\text{sup}}(i) = \mu(i)$  so  $W_{\text{sup}}$  must satisfy the first part of (4.1) because  $W$  satisfies it. On the other hand, since we assumed  $\mu_{\text{sup}}(i) = \mu(i) = j$ , for any  $j'$ , we have  $u_i(W_{\text{sup}}) = \mathbf{u}_i \geq u_i^{j'}(\mathbf{p}^{j'}) \geq u_i^{j'}(\max(\mathbf{p}^{j'}, \mathbf{p}'^{j'})) = u_i^{j'}(p^{j'}(W_{\text{sup}}))$ . So  $W_{\text{sup}}$  satisfies second part of (4.1) as well.

□